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ON ONE BOUNDARY-VALUE PROBLEM FOR THE LAPLACE EQUATION

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We have derived a Fredholm-type equation of the second kind for a directional derivative problem arising in the stationary theory of heat conduction. One result of Ya. B. Lopatinskii has been refined.

Consider the problem of finding a function $u(x)$ harmonic in the domain $D \subset E^3$ by the following boundary condition:

$$\alpha \text{ grad } u + au = f . \tag{1}$$

Let us assume that the functions α, a, and f are continuously differentiable for the required number of times. For example, we assume that α is a function twice continuously differentiable on S, and a and f are functions continuously differentiable on S. Here D is a convex domain, S is a twice continuously differentiable set, and $(\alpha, \nu) = 1$ (ν is the inner unit normal to S). The boundary condition (1) implies

$$\lim_{\tau \to 0} \left\{ (\alpha (y), \text{grad } u (x)) \big|_{x=y+\tau\nu(y)} \right\} + a (y)\, u (y) = f (y) . \tag{2}$$

Following [1], let us deduce the Green formulas for a directional derivative problem. To this end, let us make use of the identity

$$\nu = \alpha + (\alpha \cdot \nu) \cdot \nu . \tag{3}$$

The validity of formula (3) follows from the transforms

$$\nu = \alpha - \nu \cdot (\alpha \cdot \nu) = \alpha - (\alpha\, (\nu, \nu) - \nu\, (\alpha, \nu)) .$$

The latter equality is satisfied identically, since $\nu^2 = 1$, $(\alpha, \nu) = 1$. Since

$$\frac{\partial u}{\partial \nu} = (\nu, \text{grad } u) = (\alpha - \nu \cdot (\alpha \cdot \nu), \text{grad } u) = (\alpha, \text{grad } u) + (\nu, \text{grad } u \cdot (\alpha, \nu)) , \tag{4}$$

we write

$$\iint_S \left(u^* \frac{\partial u}{\partial \nu} - u \frac{\partial u^*}{\partial \nu} \right) dS = \iint_S \left(u^* (\alpha, \text{grad } u) - u\, (\alpha^*, \text{grad } u^*) + (\nu, [\text{grad } (uu^*) \cdot (\alpha \cdot \nu)]) \right) dS . \tag{5}$$

Let us introduce the following notation:

$$\alpha^* = 2\nu - \alpha , \quad a^* = a + (\nu, \text{rot } (\alpha \cdot \nu)) . \tag{6}$$

Using the formula

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$$\operatorname{rot}(uu^* [\alpha\cdot\nu]) = uu^* \operatorname{rot}(\alpha\cdot\nu) + (\operatorname{grad}(uu^*)\cdot[\alpha\cdot\nu]),$$

we obtain

$$\iint_S (\nu\,(\operatorname{grad}(uu^*)\cdot[\alpha\cdot\nu]))\,dS = \iint_S ((\nu, \operatorname{rot}(uu^* [\alpha\cdot\nu])) - (\nu, u^* \operatorname{rot}(\alpha\cdot\nu)))\,dS =$$

$$= -\iint_S (\nu, uu^* \operatorname{rot}(\alpha\cdot\nu))\,dS + \iiint_D \operatorname{grad}\operatorname{rot}(uu^* [\alpha\cdot\nu])\,d\nu = -\iint_S (\nu, uu^* \operatorname{rot}(\alpha\cdot\nu))\,dS\,. \tag{7}$$

From formulas (5)–(7) we have

$$\iint_S \left(u^* \frac{\partial u}{\partial \nu} - u\frac{\partial u^*}{\partial \nu}\right) dS = \iint_S \left(u^* ((\alpha, \operatorname{grad} u) + au) - u\,((\alpha^*, \operatorname{grad} u^*) + a^* u^*)\right) dS\,. \tag{8}$$

By virtue of the second Green formula from (8) it follows that [3]

$$\iint_S \left(u^* ((\alpha, \operatorname{grad} u) + au) - u\,((\alpha^*, \operatorname{grad} u^*) + a^* u^*)\right) dS + \iiint_D \left[u^*\Delta u - u\Delta u^*\right] dV = 0\,. \tag{9}$$

Likewise, we obtain

$$\iint_S \left(u\,((\alpha, \operatorname{grad} u) + au) - \frac{1}{2}(a + a^*)\,u^2\right) dS + \iiint_D \left[u\Delta u - (\operatorname{grad} u)^2\right] dV = 0\,. \tag{10}$$

From formula (10) it follows that if $a < 0$, $a^* < 0$, and the directional derivative problem (1) for the Laplace equation has a solution, then this solution is unique [1]. Indeed, assuming the existence of two solutions u_1 and u_2, we obtain for their difference $u = u_1 - u_2$ from (10) the equality

$$-\iint_S \frac{1}{2}(a + a^*)\,u^2 dS + \iiint_D (\operatorname{grad} u)^2 dV = 0\,,$$

whence $u \equiv 0$ in the domain D follows.

We shall seek the harmonic function in the form [1]

$$u(x) = \iint_S g(x, y)\,\mu(y)\,d_y S\,, \tag{11}$$

where $\mu(y)$ is a function continuous on S, and the function $g(x, y)$, which is harmonic at $x \neq y$, is defined by the formula [1]

$$g(x, y) = -\frac{1}{\pi}\,\frac{\left(|x - y|\,\dfrac{\alpha(y)}{|\alpha(y)|} + x - y, \nu(y)\right)}{\left|\,|x - y|\,\dfrac{\alpha(y)}{|\alpha(y)|} + x - y\right|^2}\,. \tag{12}$$

Here x and y are points of a three-dimensional Euclidean space (they are identified with the corresponding radius vectors), and $|x - y|$ is the distance between them. We represent formula (12) as

$$g(x, y) = -\frac{1}{2\pi\,|x-y|}\,\frac{\dfrac{(\alpha(y), \nu(y))}{|\alpha(y)|} + (w(x, y), \nu(y))}{1 + \left(\dfrac{\alpha(y), w(x-y)}{|\alpha(y)|}\right)}, \tag{13}$$

where

$$w(x-y) = \frac{x-y}{|x-y|}. \tag{14}$$

From (13) it follows that the thus defined function $g(x, y)$ has at $x = y$ a polar singularity. Therefore, the integral in formula (11) should be considered as improper.

Direct calculations show that

$$-2\pi\frac{\partial g}{\partial x_k} = -\frac{x_k - y_k}{|x-y|^3}\,\frac{\dfrac{(\alpha(y), \nu(y))}{|\alpha(y)|} + (w(x, y), \nu(y))}{1 + \left(\dfrac{\alpha(y), w(x-y)}{|\alpha(y)|}\right)} +$$

$$+\frac{\nu_k - \dfrac{x_k - y_k}{|x-y|}(w, \nu)}{|x-y|^2\left[1 + \left(\dfrac{\alpha(y), w(x-y)}{|\alpha(y)|}\right)\right]^2} - \frac{\left(\dfrac{\alpha_k(y)}{|\alpha(y)|} - \dfrac{x_k - y_k}{|x-y|}\left(w, \dfrac{\alpha(y)}{|\alpha(y)|}\right)\right)}{|x-y|^2\left[1 + \left(\dfrac{\alpha(y), w(x-y)}{|\alpha(y)|}\right)\right]^2},$$

therefore,

$$2\pi(\alpha, \operatorname{grad} g) = -\frac{(\alpha, w)}{|x-y|^2}\,\frac{\dfrac{(\alpha(y), \nu(y))}{|\alpha(y)|} + (w(x, y), \nu(y))}{1 + \left(\dfrac{\alpha(y), w(x-y)}{|\alpha(y)|}\right)} +$$

$$+\frac{(\alpha, \nu) - (\alpha, w)(w, \nu)}{|x-y|^2\left[1 + \left(\dfrac{\alpha(y), w(x-y)}{|\alpha(y)|}\right)\right]} - \frac{\left(\dfrac{(\alpha(y), \nu(y))}{|\alpha(y)|} + (w, \nu)\right)\left(|\alpha| - \dfrac{(w, \alpha)}{|\alpha|}\right)}{|x-y|^2\left[1 + \left(\dfrac{\alpha(y), w(x-y)}{|\alpha(y)|}\right)\right]^2}.$$

Apparent simplifications lead to the formula

$$(\alpha, \operatorname{grad} g) = -\frac{1}{2\pi}\,\frac{|\alpha|\,(w, \nu)}{|x-y|^2} = -\frac{(x-y, \nu(y))\cdot|\alpha|}{|x-y|^3}. \tag{15}$$

Let $z \in S$ and $x \in D$. Then the integral in (11) admits differentiation under the integral sign and manipulations yield

$$(\alpha(z), \operatorname{grad} u(x)) + a(z)\,u(x) =$$

$$= \iint_S \left\{((\alpha(z) - \alpha(y)), \operatorname{grad} g(x, y)) + a(z)\,g(x, y) + \frac{1}{2\pi}\,\frac{(x-y, \nu(y))\,|\alpha|}{|x-y|^3}\right\}\mu(y)\,d_yS. \tag{16}$$

If the function $\mu(y)$ is continuous on S, then the integral in (16) converges for all z. For function (11) to satisfy the boundary condition (1), one has to require that the function $\mu(y)$ be a solution of some integral equation. From the definition of $g(x, y)$ and formulas (13) and (15) it follows that the chief singularity of the subintegral expression in (16) is contained in the last term. Therefore, let us consider the integral

$$I = \iint_S \frac{(x-y, \nu(y)) \cdot |\alpha|}{|x-y|^3} \mu(y)\, d_y S . \tag{17}$$

Let us single out on S a point z_0 and denote by S_0 a small vicinity of this point, whose projection on the tangent plane forms a circle of radius ε. For this projection, let us assume that the origin of the coordinate system is at the point z_0, the y_1- and y_2-axes lie in the tangent plane, the y_3-axis is normal to S, and the equation for S_0 is representable in the form $y_3 = \omega(y_1, y_2)$, where ω is a twice continuously differentiable function, with

$$\left|\omega(y_1, y_2)\right| \le A\left(y_1^2 + y_2^2\right), \quad \left|\frac{\partial}{\partial y_i}\omega\right| \le A\sqrt{y_1^2 + y_2^2}, \quad i = 1, 2,$$

here A is some constant. Let us transform integral (17) to the form

$$I = \iint_{S_0} \frac{(x-y, \nu(y)) \cdot |\alpha|}{|x-y|^3} \mu(y)\, d_y S + \iint_{S/S_0} \frac{(x-y, \nu(y)) \cdot |\alpha|}{|x-y|^3} \mu(y)\, d_y S =$$

$$= \iint_{T_0} \frac{(x-y, \nu(y)) \cdot |\alpha|}{|x-y|^3} (\mu(y) - \mu(z_0)) \sqrt{1 + \left(\frac{\partial \omega}{\partial y_1}\right)^2 + \left(\frac{\partial \omega}{\partial y_2}\right)^2}\, dy_1 dy_2 +$$

$$+ \iint_{T_0} \frac{(x-y, \nu(y)) \cdot |\alpha|}{|x-y|^3} \mu(z_0) \sqrt{1 + \left(\frac{\partial \omega}{\partial y_1}\right)^2 + \left(\frac{\partial \omega}{\partial y_2}\right)^2}\, dy_1 dy_2 + \iint_{S/S_0} \frac{(x-y, \nu(y)) \cdot |\alpha|}{|x-y|^3} \mu(y)\, d_y S . \tag{18}$$

From (18) it is seen that the principal part I is represented by the second integral on the right side. To investigate its behavior at $x \to z_0 \in S_0$, assume that in the above local coordinate system x has coordinates $(0, 0, x_3)$, y — $(\rho \cos \varphi, \rho \sin \varphi, 0)$, $\nu(y)$ lies on the z-axis and has coordinates $(0, 0, 1)$. Therefore, for the principal part I we write

$$I = \mu(z_0) \int_0^{2\pi} \int_0^{\varepsilon} \frac{|\alpha(z_0)\, x_3|}{\left(\rho^2 + x_3^2\right)^{3/2}} \rho\, d\rho\, d\varphi + I_0(x, y) , \tag{19}$$

where $I_0(x, y)$ is the regular part of I. The integral appearing in (19) is calculated explicitly, and for I the representation

$$I = -2\pi \left|\alpha(z_0)\right| \mu(z_0)\, x_3 \left[\frac{1}{\sqrt{\varepsilon^2 + x_3^2}} - \frac{1}{|x_3|}\right] + I_0(x, y) \tag{20}$$

is valid. When $x_3 \to 0$ in (20), we get

$$\lim_{x_3 \to 0} I = 2\pi \left|\alpha(z_0)\right| \mu(z_0) + I_0 . \tag{21}$$

Thus, introducing (11) into the boundary condition (1) and passing to the limit first at $x_3 \to 0$ and then at $\varepsilon \to 0$, we arrive at the following integral equation for determining $\mu(x)$:

$$|\alpha(y)|\ \mu(x) + \iint\limits_S \Big[((\alpha(z) - \alpha(y)), \operatorname{grad} g(x, y)) + a(x)\, g(x, y) + $$

$$ + \frac{1}{2\pi} \frac{(x - y, \nu(y))}{|x - y|^3}\ |\alpha(y)| \Big] \mu(y)\, d_y S = f(x)\,. \tag{22}$$

If S is a Lyapunov surface, then the subintegral expression in (22) has an integrable singularity and, therefore, (22) is a Fredholm equation of the second kind. In the general case, it is solvable by the third Fredholm theorem.

Note. The principal results obtained here are also valid for weaker assumptions about the initial conditions of the problem. For example, the surface S can be a Lyapunov surface, and the sought functions u and u^* can be continuously differentiable in $\bar{D} = D \cup S$ and twice continuously differentiable in D.

NOTATION

a, heat-transfer coefficient; u, temperature; α, heat conductivity coefficient.

REFERENCES

1. Ya. B. Lopatinskii, *On one boundary problem for harmonic functions, Uch. Zap. L'vovsk. Univ., Ser. Fiz.-Matem. Nauk*, **22**, Issue 5 (1953).
2. Ya. B. Lopatinskii, *Introduction to the Modern Theory of Partial-Differential Equations* [in Russian], Naukova Dumka, Kiev (1980).
3. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* [in Russian], Nauka, Moscow (1978).
4. V. I. Smirnov, *A Course in Higher Mathematics* [in Russian], Vol. 4, Nauka, Moscow (1958).